mein-physikalischer Wichtigkeit sein kann, als die Bedingungen bei der Ableitung der Maxwellschen Gleichungen aus den tieferliegenden Gleichungen des metrischen Feldes eingehalten sind. Diese Bedingungen waren [2]:

1. Die Zeit ist kontinuisierbar, was heißt, daß Prozesse, die schneller als in etwa 10^{-20} sec ablaufen, nicht erfaßt werden.
2. Die Gesamtdauer der Prozesse kommt nicht in die Größenordnung kosmogonischer Prozesse.

Die Bedingung 1. macht es fraglich, ob Lorentzinvariante Gleichungen noch Prozesse beschreiben können, an denen Elementarteilchen-Resonanzen beteiligt sind. Die Bedingung 2. läßt die Anwendung Lorentz-invarianter Gleichungen in der Kosmogonie als zweifelhaft erscheinen.

On the Quantum Theory of the Anharmonic Oscillator in Functional Space

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Dynamics of quantum field theory can be formulated by functional equations. For strong interaction nonperturbative solutions of these functional equations are required. For the investigation of solution procedures the model of an anharmonic oscillator is used, because of its structural equivalence with dressed one- and two-particel states of field theory. To perform a variational solution procedure a scalar product for the state functionals is introduced and its existence is proven. The scalar product definition admits a mapping of the physical Hilbert space on the functional space. Therefore a "functional" quantum theory seems to be possible. The whole procedure can be transferred to relativistic invariant field theories, provided these theories are regularized to give finite results at all.

In quantum field theory especially in nonlinear spinor theory of elementary particles the dynamical behaviour of the physical systems can be described by functionals of field operators in a Heisenberg representation and corresponding functional equations [1–6]. To obtain the physical information in the strong coupling case, it is necessary to solve the functional equations without perturbation theory. As has been discussed in previous papers the anharmonic oscillator is a suitable test problem for the investigation of solution procedures of strong coupling functional equations [5, 6]. Therefore in the following this model will be treated. The general idea for the nonperturbative solution of the functional equations is the use of an expansion of the physical functionals into series of suitably chosen base functionals and to approximate the exact infinite series by series with a finite number of terms [7–11]. The complete theory then requires the proof of convergence and the explicit calculation of the approximate functionals. For the limiting case of single time functionals, being ordinary functions in the oscillator model both problems have been discussed [7, 10, 12–16]. For the general time ordered functionals only the second problem has been investigated in preceding papers [8, 9]. In this paper now we attack the convergence problem. As is well known from functional analysis a necessary condition for convergence is the

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existence of a scalar product definition. Therefore the main effort is made in this direction. Guided by the idea, that the entire quantum theory should be expressible in a selfconsistent way as a functional quantum theory, we propose a certain scalar product definition and prove its existence for the exact physical state functionals. As a consequence a mapping of the physical Hilbert space on the functional Hilbert space is possible and one is able to formulate quantum theory in functional space only. Because the scalar product definition proposed is explicitly a many-time formalism one may use it for canonical as well as for noncanonical quantized theories. It is especially this feature which is of interest for nonlinear spinor theory of elementary particles, because so far no proper scalar product definition has been given there. The treatment of the problem stated here is by no means complete, so further papers with theoretical as well as numerical analysis are in preparation.

1. Fundamentals

Identifying $q(t)$ with $\psi_1(t)$ and $p(t)$ with $\psi_2(t)$ the equations of motion for an anharmonic oscillator can be written according to I in the general form

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi_\alpha(t) = B_{\alpha\beta}\,\psi_\beta(t) + C_{\alpha\beta}\,D_{\gamma\sigma}\,\psi_\beta(t)\,\psi_\gamma(t)\,\psi_\sigma(t) \tag{1.1}$$

with the commutation relation

$$[\psi_\alpha(t)\ \psi_\beta(t)]_- = i\,A_{\alpha\beta}\,\mathbf{1}\,. \tag{1.2}$$

This representation is in complete analogy to the nonlinear spinor equation of elementary particle theory with Hermitean field operators [6]. The matrices B, C, D are to a certain extent still arbitrary, while A is fixed by the canonical quantization to be

$$A_{\alpha\beta} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{1.3}$$

A very suitable anharmonic oscillator model [17] is given by the following choice $B = C = A$ and $D = \delta_{\gamma\sigma}$. Its stationary energy eigenstates are

$$|\,n\rangle := f_n(\psi_1) \tag{1.4}$$

with $\psi_1 \equiv \psi_1(0)$ and

$$f_n(x) = \pi^{-\frac{1}{4}}\,(2^n\,n!)^{-\frac{1}{2}}\,e^{-\frac{1}{2}x^2}\,H_n(x) \tag{1.5}$$

where H_n are the standard Hermitean polynomials. The corresponding energies are

$$E_n = (n+\tfrac{1}{2}) + (n+\tfrac{1}{2})^2. \tag{1.6}$$

For the field theoretic functional treatment we characterize the stationary states of the model by state functionals

$$\mathfrak{T}_a(j) := \langle 0\,|\,T\exp i\int \psi_\alpha(\xi)\,j_\alpha(\xi)\,\mathrm{d}\xi\,|\,a\rangle \tag{1.7}$$

where $j_\alpha(\xi)$ are commuting functional source-functions. Using the procedure outlined in II, App. I, we obtain for the state functionals the functional equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\frac{\delta}{i\,\delta j_\alpha(t)}\,\mathfrak{T}_a(j) = \left[C_{\alpha\beta}\,D_{\gamma\sigma}\,\frac{\delta}{i\,\delta j_\beta(t)}\,\frac{\delta}{i\,\delta j_\gamma(t)}\,\frac{\delta}{i\,\delta j_\sigma(t)} + B_{\alpha\beta}\,\frac{\delta}{i\,\delta j_\beta(t)} - A_{\alpha\beta}\,j_\beta(t)\right]\mathfrak{T}_a(j) \tag{1.8}$$

and the subsidiary condition

$$\boldsymbol{P}\,\mathfrak{T}_a(j) := \int j_\alpha(t)\,\frac{\mathrm{d}}{\mathrm{d}t}\,\frac{\delta}{i\,\delta j_\alpha(t)}\,\mathrm{d}t\,\mathfrak{T}_a(j) = -\,\omega_a\,\mathfrak{T}_a(j) \tag{1.9}$$

with $\omega_a = (E_a - E_0)$. Combining Eqs. (1.8) and (1.9) we obtain the eigenvalue equation

$$\int j_\alpha(t)\left[C_{\alpha\beta}\,D_{\gamma\sigma}\,\frac{\delta}{i\,\delta j_\beta(t)}\,\frac{\delta}{i\,\delta j_\gamma(t)}\,\frac{\delta}{i\,\delta j_\sigma(t)} + B_{\alpha\beta}\,\frac{\delta}{i\,\delta j_\beta(t)}\right]\mathrm{d}t\,\mathfrak{T}_a(j) = -\,\omega_a\,\mathfrak{T}_a(j) \tag{1.10}$$

where the last term on the right side of (1.8) drops out.

For simplicity we shall work in the following with Eq. (1.10). In nonlinear spinor theory it turns out, that this equation is not explicitly relativistically invariant and therefore one has to use (1.8) and (1.9). But all arguments applied to (1.10) are valid as well as for (1.8) and (1.9), so no restriction is imposed on the entire problem by using (1.10). For the further investigation we have to expand the functionals (1.7) into power series. We obtain

$$\mathfrak{T}_a(j) = \sum_{k=1}^{\infty}\frac{i^k}{k!}\int \tau_k\begin{pmatrix} t_1 \dots t_k \\ \alpha_1 \dots \alpha_k \end{pmatrix} j_{\alpha_1}(t_1)\dots j_{\alpha_k}(t_k)\,\mathrm{d}t_1\dots\mathrm{d}t_k \tag{1.11}$$

[17] F. Bopp, Thermostatistics, Lecture, University of Munich 1965.

with

$$\tau_k\binom{t_1 \dots t_k}{\alpha_1 \dots \alpha_k} := \langle 0 \,|\, T\, \psi_{\alpha_1}(t_1) \dots \psi_{\alpha_k}(t_k) \,|\, a \rangle \tag{1.12}$$

which follows immediately from the exponential series of (1.7).

2. Rigged Hilbert Space Representation

In the preceding section the state functionals were introduced only formally. For working with them we have to achieve their real existence. Thereby it is more convenient to define the state functionals by the expansion (1.11). Because in (1.11) the τ_k-functions are distributions we have to restrict the source functions $j_\alpha(\xi)$ in a suitable way in order to obtain at least finite values of the integrals. A special choice for the source functions is the space S of all rapidly decreasing and infinitely often differentiable functions. Introducing a complete set of orthonormalized base functions in this space, on may expand the source functions within this set. Without any loss of generality we may choose the oscillator functions (1.5) to be this set. Then we have

$$j_\alpha(t) = \sum_k \varrho_{k\alpha} f_k(t) \tag{2.1}$$

and by substituting (2.1) into (1.11) the functionals are given by

$$\mathfrak{T}(j) = \boldsymbol{T}(\varrho) := \sum_n \sum_{l_1 \dots l_n} \frac{i^n}{n!}\, T\binom{l_1 \dots l_n}{\alpha_1 \dots \alpha_n} \varrho_{l_1 \alpha_1} \dots \varrho_{l_n \alpha_n} \tag{2.2}$$

with

$$T\binom{l_1 \dots l_n}{\alpha_1 \dots \alpha_n} := \int \tau_n\binom{t_1 \dots t_n}{\alpha_1 \dots \alpha_n} f_{l_1}(t_1) \dots f_{l_n}(t_n)\, \mathrm{d}t_1 \dots \mathrm{d}t_n\,. \tag{2.3}$$

In App. II the existence and a numerical estimate of (2.3) is proven. By (2.3) one learns, that this representation means a spectral decomposition of the set (1.12) by L_2-functions. Because the functions (1.12) are distributions this spectral decomposition leads to a rigged Hilbert space representation [18] of the τ_k-functions. Using the completeness relation of the f_k-functions from (2.3) they follow to be

$$\tau_n\binom{t_1 \dots t_n}{\alpha_1 \dots \alpha_n} = \sum_{u_1 \dots u_n} T\binom{u_1 \dots u_n}{\alpha_1 \dots \alpha_n} f_{u_1}(t_1) \dots f_{u_n}(t_n)\,. \tag{2.4}$$

This representation can be proven also by direct evaluation of the matrix representation of the τ_k-functions.

For working with such a representation one has to transform also the eigenvalue equation resp. the other dynamical equations. By the completeness relations follows

$$\frac{\delta}{\delta j_\alpha(t)} = \sum_s \frac{\partial}{\partial \varrho_{s\alpha}} f_s(t) \tag{2.5}$$

and the transformed eigenvalue equation (1.10) reads

$$\left[\sum_{k_1 \dots k_4} \varrho_{k_1 \alpha_1} \sum_{j=2}^{4} \frac{\partial}{i\, \partial \varrho_{k_j \alpha_j}} F\binom{k_1 \dots k_4}{\alpha_1 \dots \alpha_4} + B_{\alpha\beta} \sum_k \varrho_{k\alpha} \frac{\partial}{i\, \partial \varrho_{k\beta}} + \omega_a \right] \boldsymbol{T}_a(\varrho) = 0 \tag{2.6}$$

with

$$F\binom{k_1 \dots k_4}{\alpha_1 \dots \alpha_4} := C_{\alpha_1 \alpha_2} D_{\alpha_3 \alpha_4} \int \prod_{j=1}^{4} f_{k_j}(t)\, \mathrm{d}t\,. \tag{2.7}$$

Additionally the state functionals have to satisfy the condition of stationarity (1.9), which reads in the rigged Hilbert space representation

$$\boldsymbol{P}\, \boldsymbol{T}_a(\omega) = -\omega_a \boldsymbol{T}_a(\omega) \tag{2.8}$$

with

$$\boldsymbol{P} := \sum_k \left[-\sqrt{\frac{k+1}{2}}\, \varrho_{k+1,\alpha} + \sqrt{\frac{k}{2}}\, \varrho_{k-1,\alpha} \right] \frac{\partial}{i\, \partial \varrho_{k\alpha}}\,. \tag{2.9}$$

In nonlinear spinor theory the decomposition (2.1) corresponds to a Lorentz invariant representation of $j_\alpha(x)$ according to the irreducible representations of the rotation group. These representations are available [19, 20]. Therefore the rigged Hilbert space concept has a meaning in field theory too and all calculations made for the anharmonic oscillator in this representation can be applied equally well in field theory.

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3. Norm Definition and Existence

Formation of a functional norm requires functional integration [21]. To avoid all difficulties connected with the definition of a continuous functional integration, we perform functional integration in the rigged Hilbert space representation. Then this integration runs over a countably infinite set of variables and can be evaluated by algebraic methods. This is performed in App. I. Naturally the results of a genuine functional integration should be the same, but we do not discuss this problem here. Because the functionals (2.2) are not simply square summable for a proper norm definition weighting factors have to be introduced. We shall use two such factors. The first factor is derived functionally by the observation that for the operator

$$\boldsymbol{N} := \sum_k \varrho_{k\alpha} \frac{\partial}{\partial \varrho_{k\alpha}} \tag{3.1}$$

the functional equation

$$\boldsymbol{N}\, \varrho_{l_1\alpha_1} \dots \varrho_{l_n\alpha_n} = n\, \varrho_{l_1\alpha_1} \dots \varrho_{l_n\alpha_n} \tag{3.2}$$

holds, i. e. the power functions of $\varrho_{l\alpha}$ with arbitrary arguments are eigenfunctions of this operator. Also any steady function of $\boldsymbol{N}$ is defined in application to this eigenvector system, and can be applied therefore to $\boldsymbol{T}(\varrho)$ too. Using this property we define the first weighting factor by

$$\Gamma^{-\nu}(\boldsymbol{N})\, \boldsymbol{T}(\varrho) = \sum_n \sum_{l_1 \dots l_n} \frac{i^n}{(n!)^{\nu+1}} T\binom{l_1 \dots l_n}{\alpha_1 \dots \alpha_n} \varrho_{l_1\alpha_1} \dots \varrho_{l_n\alpha_n} \tag{3.3}$$

where the value of ν being a positive integer remains still open. The second weighting factor is defined by a generalized Gaussian weight function

$$\exp\{-\tfrac{1}{2} \Sigma\, \varrho_{n\alpha}\, G_{n\alpha\, m\beta}\, \varrho_{m\beta}\}$$

as is common to functional integration. By means of these two weighting factors we define the functional norm of $\boldsymbol{T}(\varrho)$ by the prescription

$$\| \boldsymbol{T}(\varrho) \|^2 := \int \exp\{-\Sigma\, \varrho_{n\alpha} (\mathfrak{Re}\, G_{n\alpha\, m\beta})\, \varrho_{m\beta}\} \, | \Gamma^{-\nu}(\boldsymbol{N})\, \boldsymbol{T}(\varrho)|^2 \, \mathrm{d}\tau(\varrho) \tag{3.4}$$

where $\mathrm{d}\tau(\varrho)$ means simply Riemannian integration over all ϱ-variables. Again the numerical values of $G_{n\alpha\, m\beta}$ are still open. In the next section we try to fix these values by physical considerations. In this section we prove only under very general assumptions about $G_{n\alpha\, m\beta}$ the existence of (3.4). To avoid unnecessary complication of the formulas, we introduce superindices, i. e. we denote the spinorial indices α and the expansion indices k of (2.1) by only one superindex. Then the norm formula (3.4) reads explicitly

$$\| \boldsymbol{T}(\varrho) \|^2 = \sum_{k,l} \sum_{\substack{s_1 \dots s_k \\ t_1 \dots t_l}} T_k^{\times}(s_1 \dots s_k) \frac{(-i)^k}{(k!)^{\nu}} \langle D_k(s_1 \dots s_k, G)\, D_l(t_1 \dots t_l, G) \rangle \frac{i^l}{(l!)^{\nu}} T_l(t_1 \dots t_l) \tag{3.5}$$

where the Dyson functionals [8] D_k in superindices are defined by

$$D_k(h_1 \dots h_k\, G) := \frac{1}{k!}\, \varrho_{h_1} \dots \varrho_{h_k} \exp\{-\tfrac{1}{2} \Sigma\, \varrho_n\, G_{nm}\, \varrho_m\} \tag{3.6}$$

and the brackets symbolize functional integration. In App. I the following formula for the functional integral of D_k and D_l is derived

$$\langle D_k(s_1 \dots s_k, G)\, D_l(t_1 \dots t_l, G) \rangle = \frac{\det U}{k!\, l!} \sum_{r=0}^{\min(k,l)}{}' C_r^{kl} P \sum_{\substack{\lambda_1 \dots \lambda_k \\ \varepsilon_1 \dots \varepsilon_l}} \prod_{i=1}^{r} R s_{\lambda_i} t_{\varepsilon_i} \prod_{i=\frac{1}{2}r+1}^{\frac{1}{2}k} R s_{\lambda_{2i}} s_{\lambda_{2i-1}} \prod_{i=\frac{1}{2}r+1}^{\frac{1}{2}l} R t_{\varepsilon_{2i}} t_{\varepsilon_{2i-1}} \tag{3.7}$$

where R is equal to $(\mathfrak{Re}\, G)^{-1}$ and the matrix U is connected with the spectral representation of $\mathfrak{Re}\, G$. Details are given in App. I. From App. I follows, that $\mathfrak{Re}\, G$ has to have a spectral representation with all eigenvalues unequal zero. Then $(\mathfrak{Re}\, G)^{-1}$ exists also.

Substituting (3.7) into (3.5) and observing the permutational symmetries of the T_k-expansion coefficients (3.5) becomes

$$\| \boldsymbol{T}(\varrho) \|^2 = \det U \sum_{k,l} \sum_{\substack{s_1 \dots s_k \\ t_1 \dots t_l}} \sum_r (-1)^k \frac{C_r^{kl}\, i^{k+l}}{(k!\, l!)^{\nu}} T_k^{\times}(s_1 \dots s_l)\, T_l(t_1 \dots t_l)\, R_{s_1 t_1} \dots R_{s_r t_r} R_{s_{r+1} s_{r+2}} \dots R_{t_{l-1} t_l} . \tag{3.8}$$

For the further evaluation of (3.8) we have to make an estimate of T_k. This is done in App. II. One obtains

$$\left| T_n\binom{u_1 \dots u_n}{\alpha_1 \dots \alpha_n} \right| \leqq n^{3/2} (n!)^{3/2} C^n . \tag{3.9}$$

[21] K. FRIEDRICHS and A. SHAPIRO, Seminar on Integration of Functionals, New York University.

By this approximation (3.8) goes over into the inequality

$$\| \boldsymbol{T}(\varrho) \|^2 \leqq \sum_{k,l} \sum_{r=0}^{\min(k,l)} C_r^{kl} \frac{(k\,l)^{3/2}}{(k!\,l!)\,\nu^{-3/2}} K^{(k+l)} \,|\det U| \tag{3.10}$$

with

$$K := \Big(\sum_{uu'} |R_{uu'}|\Big)^{\frac{1}{2}} C \tag{3.11}$$

and according to App. I

$$C_r^{kl} = \frac{1}{\left(\frac{k-r}{2}\right)!\left(\frac{l-r}{2}\right)! r!} \left(\tfrac{1}{2}\right)^{k+l-r}. \tag{3.12}$$

Therefore

$$\| \boldsymbol{T}(\varrho) \|^2 \leqq \sum_{k,l} \sum_{r=0}^{\min(k,l)} \frac{K^{(k+l)}\,(k\,l)^{3/2}}{\left(\frac{k-r}{2}\right)!\left(\frac{l-r}{2}\right)!\, r!\,(k!\,l!)^{\nu-3/2}} \left(\tfrac{1}{2}\right)^{k+l-r} |\det U| \tag{3.13}$$

is obviously convergent as long as (3.11) is finite, $\det U$ exists and $\nu \geqq \frac{5}{2}$. Detailed calculations show, that the finiteness of (3.11) and of $\det U$ is probably a contradiction. So one would like to suppress $\det U$. This is easily possible, because $\det U$ plays the role of a renormalisation constant being for all integrals the same. It can be eliminated by defining the renormalized norm

$$\| \boldsymbol{T}_a(\varrho) \|_{\text{ren.}} := \frac{\| \boldsymbol{T}_a(\varrho) \|}{\| \boldsymbol{T}_0(\varrho) \|} \tag{3.14}$$

where $\boldsymbol{T}_0(\varrho)$ is the ground state functional. Then the renormalized norm is a completely finite number, provided the above mentioned conditions are satisfied. Still there remains a great arbitrariness concerning the choice of the weighting factors. This we try to remove in the next section.

4. Functional Scalar Products

For developping a quantum theory in functional space, we do not only need a norm definition for the state functionals themselves but also a functional scalar product between different state functionals. Such a scalar product can be given by a straight forward extension of the norm definition (3.4) resp. (3.14). Defining the weighted state functionals by

$$\boldsymbol{W}_a(\varrho) := \exp\{-\tfrac{1}{2}\varrho\, G\, \varrho\}\, \Gamma^{-\nu}(\boldsymbol{N})\, \boldsymbol{T}_a(\varrho) \tag{4.1}$$

where $\varrho\, G\, \varrho$ is a symbolic notation of the exponential in (3.4), the scalar product between two different state functionals $\boldsymbol{T}_a$ and $\boldsymbol{T}_b$ can be written

$$\langle \boldsymbol{T}_a(\varrho)\, \boldsymbol{T}_b(\varrho) \rangle_{\text{ren.}} := \| \boldsymbol{T}_0(\varrho) \|^{-2} \int \boldsymbol{W}_a^{+}(\varrho)\, \boldsymbol{W}_b(\varrho)\, \mathrm{d}\tau(\varrho). \tag{4.2}$$

The existence of (4.2) for arbitrary state functionals $\boldsymbol{T}_a$ and $\boldsymbol{T}_b$ can be proven by the same method as used in section 3 and it is not necessary to repeat these statements. Then the norm definition is only a special case of the scalar product (4.2) and all these products are finite for physical state functionals. Therefore it follows that the weighted state functionals (4.1) can be considered to be elements of a functional Hilbert space. Substituting the definition of the state functionals $\boldsymbol{T}_a$ in (4.1) this equation can be formally written

$$\boldsymbol{W}_a(\varrho) = S\,|\,a\rangle \tag{4.3}$$

where S causes a mapping of the physical Hilbert space on to the functional Hilbert space. Contrary to the scalar product of different physical states $|\,a\rangle$ and $|\,b\rangle$ in general the functional scalar product between corresponding functional states $\boldsymbol{W}_a$ and $\boldsymbol{W}_b$ is not orthogonal. This is due to the use of a weighting factor. Therefore S is in general a nonunitary similarity transformation. In this case it is convenient to introduce a second system, the socalled dual system of state functionals $\boldsymbol{T}^a(\varrho)$ satisfying the orthonormality relations

$$\langle \boldsymbol{T}^a(\varrho)\, \boldsymbol{T}_b(\varrho) \rangle_{\text{ren.}} = \delta_b^a. \tag{4.4}$$

Assuming the $\boldsymbol{T}^a(\varrho)$ to be a linear combination of the original state functionals, we have

$$\boldsymbol{T}^a(\varrho) = \sum_b c^{ab}\, \boldsymbol{T}_b(\varrho) \tag{4.5}$$

and by (4.4) follows immediately

$$c^{ab} = (\langle \boldsymbol{T}_a(\varrho)\, \boldsymbol{T}_b(\varrho) \rangle_{\text{ren.}})^{-1}. \tag{4.6}$$

According to Schmidts orthogonalization procedure (4.6) exists if the vectors $\boldsymbol{T}_a(\varrho)$ are linearly independent and normalizable. The linear independence is guaranteed by the linear independence of the $|\,a\rangle$ states. The normalization can be performed

by a proposal of WEBER [22]. Assuming for G the ansatz

$$G_{k\alpha\, m\beta} = \delta_{km}\, g_m\, \delta_{\alpha\beta} \tag{4.7}$$

the weighting factor contains just as many unknown parameters as there are normalization conditions for the different state functionals $\boldsymbol{T}_a$. Assuming the norm expression for each functional to be unity by (3.8) one obtains a set of equations for the determination of the g_m. So in the framework of the norm and scalar product definition (4.2) we obtain a well defined dual set $\boldsymbol{T}^a$ connected with a non-unitary mapping of physical Hilbert space on to functional Hilbert space.

But now probably the definition (4.2) is not the only one that can be given for the construction of a functional Hilbert space and a mapping on to physical Hilbert space. Rather one has to expect an infinite set of possible mappings and functional Hilbert spaces. This gives rise to the question if there is any privileged functional Hilbert space or not. The answer is given in section 6. There it is shown that any functional Hilbert space leads to the same quantum mechanical information provided (4.4) is satisfied, i. e. the similarity transformation has to be a bijective one. So from the general physical point of view all possible functional Hilbert spaces are equivalent. This is true as long as one operates within the frame work of quantum mechanics. But it will be shown in another paper that the special definition of the functional scalar product becomes physically important if one treats relativistic quantum field theory. Especially the definition given here is adapted to the functional formulation of nonlinear spinor theory of elementary particles.

5. Calculational Method for State Functionals

Being able to define a weighted norm for the state functionals, one would like to use this for the approximate calculation of these functionals. A method working with weighted norms is the method of least squares. This method has been applied successfully already in quantum mechanics and for Bethe-Salpeter problems [23, 24]. Unfortunately no rigorous proof of this method is known to the author, but one learns from quantum mechanics that for an application of this method the operator should be Hermitean at least and should have a lower limit of its spectrum. Now the eigenvalue equation (2.6) is formally Hermitean and its spectrum contains with the eigenvalue ω the eigenvalue $-\omega$ too, i. e. the spectrum is invariant against inversion at $\omega = 0$. Therefore by squaring of the operator no information is lost and the spectrum has a lower bound. So the conditions being supposed for the successful application of this method are satisfied for this problem. Writing the eigenvalue equation (2.6) in the symbolic form

$$\left[\boldsymbol{O}\left(\varrho, \frac{\partial}{\partial \varrho}\right) + \omega\right] \boldsymbol{T}(\varrho) = 0 \tag{5.1}$$

the condition of variational stationarity reads in this method

$$\left\| \left[\boldsymbol{O}\left(\varrho, \frac{\partial}{\partial \varrho}\right) + \omega\right] \boldsymbol{T}(\varrho) \right\|^2 = \min . \tag{5.2}$$

This is always a positive expression. Therefore sequences of trial functions should give eigenvalues converging towards the exact values if the number of variational parameters is enlarged. For trial functions we use the approximate functionals

$$\boldsymbol{T}_N(\varrho) := \sum_{n=1}^{N} \sum_{l_1 \ldots l_n} \frac{i^n}{n!}\, T_n\begin{pmatrix} l_1 \ldots l_n \\ \alpha_1 \ldots \alpha_n \end{pmatrix}, N\Big)\, \varrho_{l_1 \alpha_1} \ldots \varrho_{l_n \alpha_n} \tag{5.3}$$

where the free variational parameters are given by the expansion coefficients $T_n(l_1 \ldots l_n, N)$. To evaluate the equations for the trial functions (5.3) one derives first the exact equations for the state functionals from (5.2) and truncates afterwards these equations. We give only a short review what has to be done. Introducing superindices (5.1) can be written

$$\sum_{m,n} \sum_{\substack{k_1 \ldots k_n \\ s_1 \ldots s_n}} T_m(s_1 \ldots s_m) O_{mn}(s_1 \ldots s_m, k_1 \ldots k_n) \frac{i^n}{n!}\, \varrho_{k_1} \ldots \varrho_{k_n} = 0 \tag{5.4}$$

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[24] K. LADÁNYI, Nuovo Cim. **56**, 173 [1968].

with

$$O_{mn}(s_1 \dots s_m, k_1 \dots k_n) := -n\,\delta_{s_1 k_1} \dots \delta_{s_{n-1} k_{n-1}} F(s_n k_n s_{n+1} s_{n+2})\,\delta_{m,n+2} + n\,\delta_{s_1 k_1} \dots \delta_{s_{n-1} k_{n-1}} B(s_n k_n)\,\delta_{mn} + \omega\,\delta_{s_1 k_1} \dots \delta_{s_n k_r} \delta_{mn}\,. \tag{5.5}$$

This representation follows from (2.6) by direct evaluation. It is the "integral" representation of (2.6). By means of (5.4) and (5.5) the minimum expression (5.2) can be written

$$\sum_{m,j} \sum_{\substack{s_1' \dots s_j' \\ s_1 \dots s_m}} T_j^{\times}(s_1' \dots s_j')\, M_{jm}(s_1' \dots s_j', s_1 \dots s_m)\, T_m(s_1 \dots s_m) = \min \tag{5.6}$$

with

$$M_{jm}(s_1' \dots s_j'\, s_1 \dots s_m) = \sum_{n,l} \sum_{\substack{k_1 \dots k_n \\ k_1' \dots k_l'}} O_{jl}^{\times}(s_1' \dots s_j'\, k_1' \dots k_l')\, \frac{(-i)^l}{(l!)^{\nu}} \langle D_l(k_1' \dots k_l'\, G)\, D_n(k_1 \dots k_n\, G) \rangle\, \frac{i^n}{(n!)^{\nu}}\, O_{nm}(s_1 \dots s_m, k_1 \dots k_n)\,. \tag{5.7}$$

Then we consider T and $T^{\times}$ to be independent quantities and vary independently both of them. This gives

$$\sum_{m} \sum_{s_1 \dots s_m} M_{jm}(s_1' \dots s_j', s_1 \dots s_m)\, T_m(s_1 \dots s_m) = 0 \tag{5.8}$$

and the conjugate complex equation, which brings no new information. As the scalar product of weighted Dyson functionals has been derived in App. I the expression of M_{jm} can be calculated explicitly without any difficulty. It is not the intention of this paper to discuss approximate variational solutions of (5.8). Thus we do not evaluate the technic of solution in detail. It runs on the same pattern like in II and III.

6. Functional Quantum Theory

In the preceding sections we discussed al those operations which seem to be necessary to constitute a functional quantum theory i. e. a theory in functional space which is an isomorphism of ordinary quantum theory. To decide whether we have really obtatined a functional quantum theory or not we consider the amount of physical information that can be derived by the functional methods and compare it with the maximum amount of information following from ordinary quantum theory. According to quantum theory the information is given by a complete set of quantum numbers, i. e. diagonalized observables and by the probability of the system to be in a certain state characterized by these numbers. Now for the anharmonic oscillator the only quantum number is the energy. It can be calculated either by conventional Schrödinger theory, or by functional calculus with equations (1.10) provided we have a powerful calculation method what we shall assume for this discussion. Therefore concerning the quantum numbers the statements of ordinary quantum theory and functional theory are the same and we have to look only for an equivalent probability definition in functional space.

To do this we remind of the probability definition in ordinary physical Hilbert space. In this space the most general state is defined by

$$\Psi(0) = \sum_a c_a\, \Psi_a(0) \tag{6.1}$$

and the occupation probability of a certain state $|a\rangle \equiv \Psi_a(0)$ is given by

$$P_a(0) = |c_a|^2 = \langle \Psi(0)\ \Psi_a(0) \rangle^2. \tag{6.2}$$

To imitate this in functional space we consider the generalized functionals

$$\mathfrak{T}(j) := \langle 0 \,|\, T \exp i \int \psi_a(\xi)\, j_a(\xi)\, \mathrm{d}\xi \,|\, \Psi(0) \rangle. \tag{6.3}$$

Substitution of (6.1) into (6.3) then gives

$$\mathfrak{T}(j) = \sum_a c_a\, \mathfrak{T}_a(j) \tag{6.4}$$

where $\mathfrak{T}_a(j)$ are the state functionals defined in (1.7), which is the analogue to (6.1). Transition to the rigged Hilbert representations changes (6.4) into

$$\boldsymbol{T}(\varrho) = \sum_a c_a\, \boldsymbol{T}_a(\varrho)\,. \tag{6.5}$$

Assuming now the definition of a scalar product under the general conditions of section 4 and the existence of a corresponding dual set $\boldsymbol{T}^a$ from (6.5) follows by using (4.4)

$$P_a(0) = |c_a|^2 = \langle \boldsymbol{T}(\varrho)\ \boldsymbol{T}^a(\varrho) \rangle_{\mathrm{ren}}^2\,. \tag{6.6}$$

By this we see, that the probability statements of ordinary quantum theory can be reproduced in functional space. Therefore all physical information to

be obtainable from ordinary quantum theory can be reproduced by functional theory. Unfortunately the functional space is larger than the physical Hilbert space, and at present no selection principle between the desired and the undesired functionals is known. Nevertheless for the quantum theory of infinite degrees of freedom one may hope, that it is not necessary in future to work with the physical Hilbert space at all and quantum theory can be formulated independently in functional space only.

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Appendix I

Already in III, App. I, II we discussed functional integration by means of the Hermitean functionals [21]. But the discussion given there is not very instructive because by the mathematical methods used in III the real structure of the problem is not revealed. A better structural understanding can be achieved by observing the isomorphism of the states for free fields with the Hermitean functionals. Although both quantities have a different meaning in connection with the present problem their formal definition is the same. As we perform our calculation in a L_2-representation we introduce from the beginning the L_2-variables, namely the ϱ_k. By means of the ϱ_k the following creation and destructions operators can be defined

$$A_k = 2^{-\frac{1}{2}}\left(\varrho_k + \frac{\partial}{\partial \varrho_k}\right), \qquad A_k^+ = 2^{-\frac{1}{2}}\left(\varrho_k - \frac{\partial}{\partial \varrho_k}\right). \tag{I.1}$$

Obeying the commutation relations

$$[A_k A_{k'}^+]_- = \delta_{kk'}\,\mathbf{1} \tag{I.2}$$

all other commutators vanish. Then the standard Hermitean functionals are defined by

$$J_n(k_1 \ldots k_n) := \frac{1}{\sqrt{n!}} A_{k_1}^+ \ldots A_{k_n}^+ \exp - \tfrac{1}{2}\sum \varrho_k^2 \tag{I.3}$$

and due to

$$A_k \exp - \tfrac{1}{2}\sum \varrho_k^2 \equiv A_k \,|\, 0\rangle = 0 \tag{I.4}$$

the functional integral between two standard Hermitean functionals

$$\langle J_l(k_1 \ldots k_l)\; J_n(k_1' \ldots k_n')\rangle := \int J_l(k_1 \ldots k_l)\; J_n(k_1' \ldots k_n') \prod_k \mathrm{d}\varrho_k \tag{I.5}$$

can be evaluated by the usual field theoretic methods giving

$$\langle J_l(k_1 \ldots k_l)\; J_n(k_1' \ldots k_n')\rangle = \delta_{ln}\frac{1}{n!}\sum_{\lambda_1 \ldots \lambda_n}^{P} \delta(k_1 - k_{\lambda_1}') \ldots \delta(k_n - k_{\lambda_n}'). \tag{I.6}$$

As long as the commutation relations (I.2) are satisfied these formulas are valid for usual indices as well as for superindices. Now from the definition of the functional variables $\varrho_{k\alpha}$ appearing in (2.1) it follows, that (I.2) is satisfied for superindices too. Therefore in the following all formulas can be read in indices as well as in superindices.

Further the connection between Dyson functionals and Hermite functionals is of interest. To derive it we introduce auxiliary variables x_k. Then by (I.4) we have

$$\exp\{\Sigma\, x_k A_k^+\}\,|\,0\rangle = \exp\{\sum_n x_k A_k^+\}\exp\{\sum_n x_k A_k\}\,|\,0\rangle \tag{I.7}$$

and by application of the Hausdorff-formula on the right side of (I.7) follows

$$\exp\{2^{\frac{1}{2}}\Sigma\, x_k \varrho_k\}\,|\,0\rangle = \exp\{\Sigma\, x_k A_k^+\}\,|\,0\rangle \exp\{\tfrac{1}{2}\sum_{kk'} x_k \delta_{kk'} x_{k'}\}. \tag{I.8}$$

Defining the standard Dyson functionals by

$$D_n(k_1 \ldots k_n) := \frac{1}{n!}\varrho_{k_1} \ldots \varrho_{k_n}\,|\,0\rangle \tag{I.9}$$

power series expansion in x_k and comparison of equal coefficients on both sides of (I.8) gives by observing (I.3) and (I.9)

$$D_n(k_1 \ldots k_n) = \frac{1}{n!}\sum_{\lambda_1 \ldots \lambda_n}^{P}\sum_{\mu=1}^{n} (\tfrac{1}{2})^{n-\frac{1}{2}\mu}\frac{1}{\sqrt{\mu!}\left(\frac{n-\mu}{2}\right)!} J_\mu(k_{\lambda_1} \ldots k_{\lambda_\mu})\,\delta_{k_{\lambda_{\mu+1}} k_{\lambda_{\mu+2}}} \ldots \delta_{k_{\lambda_{n-1}} k_{\lambda_n}}. \tag{I.10}$$

(I.10) can be used for evaluating the scalar product of two Dyson functionals. This is not done here, because we are interested in more general definitions of Hermitean- and Dyson-functionals. They can be defined by introducing

$$|0\rangle_{\mathrm{w}} := \exp\{-\tfrac{1}{2}\sum_{kl} \varrho_k a_{kl} \varrho_l\} \tag{I.11}$$

and the corresponding creation and destruction operators

$$A_j(a) := 2^{-\frac{1}{2}}\Big(\sum_l a_{jl}\varrho_l + \frac{\partial}{\partial \varrho_j}\Big),$$
$$A_j^+(a) := 2^{-\frac{1}{2}}\Big(\sum_l a_{jl}\varrho_l - \frac{\partial}{\partial \varrho_j}\Big). \tag{I.12}$$

a_{kl} is assumed to be real and symmetric. Then the "weighted" Hermitean functionals are defined by

$$J_n(k_1 \ldots k_n\, a) := \frac{1}{\sqrt{n!}} A_{k_1}^+(a) \ldots A_{k_n}^+(a)\,|0\rangle_{\mathrm{w}} \tag{I.13}$$

and their scalar product is given by

$$\langle J_n(k_1' \ldots k_n'\, a)\; J_n(k_1 \ldots k_n\, a)\rangle = \frac{1}{n!}\int A_{k_1'}^+(a) \ldots A_{k_n'}^+(a)\,|0\rangle_{\mathrm{w}}\, A_{k_1}^+(a) \ldots A_{k_n}^+(a)\,|0\rangle_{\mathrm{w}} \prod \mathrm{d}\varrho\,. \tag{I.14}$$

To evaluate it we perform a similarity transformation

$$\varrho_k = \sum_r U_{kr}\, s_r \tag{I.15}$$

where U_{kr} is defined by

$$\sum_{kl} U_{rk}^T a_{kl} U_{ll}^{-1} = \delta_{lt} \tag{I.16}$$

the volume element of the ϱ-space is transformed into

$$\prod \mathrm{d}\varrho_k = \det U \prod \mathrm{d}s_l \tag{I.17}$$

and the creation operators (I.12) can be written

$$A_j^+(a) = 2^{-\frac{1}{2}} \sum_r U_{jr}^{-1T}(s_r - \partial/\partial s_r). \tag{I.18}$$

Then the general scalar product (I.14) can be expressed due to (I.15), (I.16), (I.17), (I.18) by the standard product (I.5)

$$\langle J_n(k_1' \ldots k_n'\, a)\; J_n(k_1 \ldots k_n\, a)\rangle = \sum_{\substack{l_1' \ldots l_n' \\ l_1 \ldots l_n}} U_{k_1' l_1'}^{-1T} \ldots U_{k_n' l_n'}^{-1T}\, U_{k_1 l_1}^{-1T} \ldots U_{k_n l_n}^{-1T} \det U \langle J_n(l_1' \ldots l_n')\; J_n(l_1 \ldots l_n)\rangle. \tag{I.19}$$

Observing (I.6) and

$$a_{kl} = \sum_r U_{kr}^{-1T} U_{rl}\,. \tag{I.20}$$

(I.19) finally is given by

$$\langle J_n(k_1' \ldots k_n'\, a)\; J_n(k_1 \ldots k_n\, a)\rangle = \frac{1}{n!} {}^{P}\sum_{\lambda_1 \ldots \lambda_n} a_{k_1' k_{\lambda_1}} \ldots a_{k_n' k_{\lambda_n}} \det U\,. \tag{I.21}$$

In order to derive the connection between "weighted" Dyson functionals and weighted Hermite functionals, we introduce again auxiliary variables x_k. Observing

$$A_j^+(a)\,|0\rangle_{\mathrm{w}} = 0 \tag{I.22}$$

and

$$[A_j(a)\; A_k^{+\prime}(a)]_- = a_{jk}\,\mathbf{1}\,, \tag{I.23}$$

we obtain in analogy to (I.7) and (I.8) the relation

$$\exp\{2^{\frac{1}{2}}\sum_{kl} x_k a_{kl} \varrho_l\}\,|0\rangle_{\mathrm{w}} = \exp\{\sum_k x_k A_k^+(a)\}\,|0\rangle_{\mathrm{w}} \cdot \exp\{\tfrac{1}{2}\sum_{kl} x_k a_{kl} x_l\}\,. \tag{I.24}$$

Defining the weighted Dyson functional by

$$D_n(k_1 \ldots k_n\, a) := \frac{1}{n!}\, \varrho_{k_1} \ldots \varrho_{k_n}\,|0\rangle_{\mathrm{w}} \tag{I.25}$$

power series expansion in x_k and comparison of equal coefficients on both sides of (I.24) gives by substitution of (I.13) and (I.25)

$$D_n(l_1 \ldots l_n\, a) = \frac{1}{n!} \sum_{\lambda_1 \ldots \lambda_n} \sum_{\mu=1}^{n} \sum_{k_1 \ldots k_\mu} (\tfrac{1}{2})^{n-\frac{1}{2}\mu} \frac{1}{\sqrt{\mu!}\,\left(\frac{n-\mu}{2}\right)!}\, a_{l_{\lambda_1} k_1}^{-1} \ldots a_{l_{\lambda_\mu} k_\mu}^{-1} J_\mu(k_1 \ldots k_\mu\, a)\; a_{l_{\lambda_{\mu+1}} l_{\lambda_{\mu+2}}}^{-1} \ldots a_{l_{\lambda_{n-1}} l_{\lambda_n}}^{-1}. \tag{I.26}$$

By means of (I.26) and (I.21) the scalar product between two weighted Dyson functionals follows immediately. We obtain

$$\langle D_n(l_1 \dots l_n\, a)\; D_m(l_1' \dots l_m'\, a)\rangle$$
$$= \frac{\det U}{n!\, m!} \; P\sum_{\lambda_1 \dots \lambda_n} \; \sum_{\mu=1}^{\min(n,m)} C_\mu^{nm}\; a^{-1}_{l_{\lambda_1} l'_{\lambda'_1}} \dots a^{-1}_{l_{\lambda_\mu} l'_{\lambda'_\mu}}\; a^{-1}_{l_{\lambda_{\mu+1}} l_{\lambda_{\mu+2}}} \dots a^{-1}_{l_{\lambda_{n-1}} l_{\lambda_n}}\; a^{-1}_{l'_{\lambda'_{\mu+1}} l'_{\lambda'_{\mu+2}}} \dots a^{-1}_{l'_{\lambda'_{m-1}} l'_{\lambda'_m}} \tag{I.27}$$

with
$$C_\mu^{nm} := (\tfrac{1}{2})^{m+n-\mu}\; \frac{1}{\mu!\left(\frac{n-\mu}{2}\right)!\left(\frac{m-\mu}{2}\right)!}. \tag{I.28}$$

The scalar product between two standard Dyson functionals obviously is a special case of (I.27).

Appendix II

For an estimate of (2.3) we make use of the definition

$$T_n\begin{pmatrix} u_1 \dots u_n \\ \alpha_1 \dots \alpha_n \end{pmatrix} := \int \tau_n\begin{pmatrix} t_1 \dots t_n \\ \alpha_1 \dots \alpha_n \end{pmatrix} f_{u_1}(t_1) \dots f_{u_n}(t_n)\, \mathrm{d}t_1 \dots \mathrm{d}t_n \tag{II.1}$$

and of the spectral representation of the τ-functions by intermediate states

$$\tau_n\begin{pmatrix} t_1 \dots t_n \\ \alpha_1 \dots \alpha_n \end{pmatrix} = P\sum_{\nu_1 \dots \nu_n} \; \sum_{m_1 \dots m_{n-1}} \langle 0 \,|\, \psi_{\alpha_1} \,|\, m_1\rangle \dots \langle m_{n-1} \,|\, \psi_{\alpha_n} \,|\, a\rangle$$
$$\cdot \exp\{-i \sum_{s=1}^{n} (\omega_{m_s} - \omega_{m_{s-1}})\, t_{\nu_s}\}\, \Theta(t_{\nu_1} - t_{\nu_2}) \dots \Theta(t_{\nu_{n-1}} - t_{\nu_n}) \tag{II.2}$$

with $m_n = a$ in the exponential. By substitution of (II.2) into (II.1) follows

$$T_n\begin{pmatrix} u_1 \dots u_n \\ \alpha_1 \dots \alpha_n \end{pmatrix} = P\sum_{\nu_1 \dots \nu_n} \; \sum_{m_1 \dots m_{n-1}} \langle 0 \,|\, \psi_{\alpha_1} \,|\, m_1\rangle \dots \langle m_{n-1} \,|\, \psi_{\alpha_n} \,|\, m_n\rangle\; U(\omega_{m_1} - \omega_{m_0}, \dots, \omega_{m_n} - \omega_{m_{n-1}}, u_{\nu_1} \dots u_{\nu_n}) \tag{II.3}$$

with

$$U(x_1 \dots x_n, u_1 \dots u_n) := \int \exp\{-i \sum_{s=1}^{n} x_s\, t_s\} \Theta(t_1 - t_2) \dots \Theta(t_{n-1} - t_n)\; f_{u_1}(t_1) \dots f_{u_n}(t_n)\, \mathrm{d}t_1 \dots \mathrm{d}t_n. \tag{II.4}$$

Now from (II.4) follows the inequality

$$|\, U(x_1 \dots x_n, u_1 \dots u_n)| \leqq \prod_{j=1}^{n} \int |\, f_{u_j}(t_j)|\, \mathrm{d}t_j \tag{II.5}$$

and for the integrals an estimate can be given

$$\int |\, f_u(t)|\, \mathrm{d}t \leqq C \tag{II.6}$$

with a finite constant C for arbitrary u by means of an inequality for Hermitean polynomials [25]. Therefore (II.4) is uniformly bounded over the whole range of the variables $x_1 \dots x_n$, $u_1 \dots u_n$ and satisfies the inequality

$$|\, U(x_1 \dots x_n, u_1 \dots u_n)| \leqq C^n \tag{II.7}$$

for all n. For an estimate of the first part in (II.3) we first observe, that according to the properties of matrixelements for the oscillator functions (1.5) the equation

$$\sum_{m_1 \dots m_{n-1}=0}^{\infty} |\langle 0 \,|\, \psi_{\alpha_1} \,|\, m_1\rangle| \dots |\langle m_{n-1} \,|\, \psi_{\alpha_n} \,|\, m_n\rangle| \tag{II.8}$$
$$= \sum_{\varkappa=2}^{n-1} \sum_{m\varkappa=1}^{\varkappa} |\langle 0 \,|\, \psi_{\alpha_1} \,|\, 1\rangle|\, |\langle 1 \,|\, \psi_{\alpha_2} \,|\, m_2\rangle| \dots |\langle m_{n-1} \,|\, \psi_{\alpha_n} \,|\, m_n\rangle|$$

holds. Disregarding all constants we have further the inequality

$$|\langle m_{\nu-1} \,|\, \psi_{\alpha_\nu} \,|\, m_\nu\rangle| \leqq \sqrt{\nu}, \qquad m_{\nu-1} = 1 \dots \nu - 1 \tag{II.9}$$

and combining (II.8) and (II.9) it follows

$$\sum_{m_1 \dots m_{n-1}} |\langle 0 \,|\, \psi_{\alpha_1} \,|\, m_1\rangle| \dots |\langle m_{n-1} \,|\, \psi_{\alpha_n} \,|\, m_n\rangle| \leqq n^2 (n-1)!^{\frac{1}{2}}. \tag{II.10}$$

By substituting (II.10) and (II.7) into (II.3) we finally obtain

$$\left| T_n\begin{pmatrix} u_1 \dots u_n \\ \alpha_1 \dots \alpha_n \end{pmatrix} \right| \leqq n^{3/2} (n!)^{3/2} C^n \tag{II.11}$$

being the desired estimate for T_n.

[25] I. M. Ryshik and I. S. Gradstein, Tafeln, Deutscher Verb. d. Wiss., Berlin 1957, ge. 7.124.